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Three pillars pension provision system

1. public (state) pension
2. collective pension funds
   - defined benefit (DB) pension schemes
   - defined contribution (DC) pension schemes
3. individual saving

Accumulation vs decumulation

The actuarial literature on the financial risk in DC schemes can be divided in two main stream:

1. research on the investment strategy to be adopted in the accumulation phase, i.e. between joining the fund and retirement
2. research on decision-making in the decumulation phase, i.e. between retirement and death, assuming the income drawdown is taken (i.e. the fund accumulates, the retiree withdraws period income and annuitizes at future time T)

Some example of optimization problems in the accumulation phase

Let n be the number of year of service, $F(n)$ be the quantity to be optimized, e.g. $F(n)$ be the fund at time n, or the replacement ratio at time n, or the pension rate at time n. Then, possible optimization problems are

- maximize $E(F(n))$
- maximize $E(F(n))$ subject to $Var(F(n)) = \sigma^2$
- minimize $Pr(F(n) < F_{MIN})$ where F_{MIN} is some minimum level of $F(n)$ needed by the member
- maximize $E(U(F(n)))$ where $U(\cdot)$ is some utility function
- minimize $E[(F(n) - F_{TAR})^2 | H_t]$ where F_{TAR} is some final target, and H_t is the history up to time t
- maximize $E[U(F(n)) | F_{EST}, H_t]$ where F_{EST} is the estimate of $F(n)$ made at time t when the information known is H_t

$\Rightarrow$ all these problems can be solved via dynamic programming techniques to find the **optimal investment strategy** according to the criterion selected.

An example

Mean-variance dynamic optimality for DC pension schemes

Joint work with Francesco Menoncin,
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General model (s state variables, n risky assets)

The financial market is described by the following variables:

- s state variables $z(t)$ (with $z(0) = z_0 \in \mathbb{R}^s$ known) whose values solve the stochastic differential equation (SDE)

$$\underset{s\times 1}{dz(t)} = \underset{s\times 1}{\mu_z}(t,z)dt + \underset{s\times n}{\Omega}(t,z)\underset{n\times 1}{dW(t)}; \tag{1}$$

- one riskless asset whose price $G(t)$ solves the (ordinary) differential equation

$$dG(t) = G(t)r(t,z)\,dt,$$

  where $r(t,z)$ is the spot instantaneously riskless interest rate;

- n risky assets whose prices $P(t)$ (with $P(0) = P_0 \in \mathbb{R}^n$ known) solve the matrix stochastic differential equation

$$\underset{n\times 1}{dP(t)} = \underset{n\times n}{I_P}\left[\underset{n\times 1}{\mu}(t,z)dt + \underset{n\times n}{\Sigma}(t,z)\underset{n\times 1}{dW(t)}\right], \tag{2}$$

  where I_P is the $n \times n$ square diagonal matrix that reports the prices $P_1, P_2, ..., P_n$ on the diagonal and zero elsewhere.

General model, wealth dynamics

The wealth dynamics are given by the following stochastic differential equation (SDE)

$$dX(t) = \big(X(t)\, r(t,z) + c(t,z) + w(t)'\,(\mu(t,z) - r(t,z)\,\mathbf{1})\big)\, dt + w(t)'\,\Sigma(t,z)\, dW(t)\,. \quad (3)$$

Particular case (stochastic salary and interest rate)

Here we assume $s = n = 2$. There are two state variables: (i) the interest rate $r(t)$ which follows a Vasicek process, and (ii) the contribution $c(t)$ which follows a GBM. Thus, the vector $z(t) = [r(t) \quad c(t)]'$ can be represented in the following way:

$$\underbrace{\begin{bmatrix} dr(t) \\ dc(t) \end{bmatrix}}_{dz(t)} = \underbrace{\begin{bmatrix} a(b - r(t)) \\ c(t)\mu_c \end{bmatrix}}_{\mu_z(t,z)} dt + \underbrace{\begin{bmatrix} \sigma_r & 0 \\ c(t)\sigma_{cr} & c(t)\sigma_{cs} \end{bmatrix}}_{\Omega(t,z)} \underbrace{\begin{bmatrix} dW_r(t) \\ dW_s(t) \end{bmatrix}}_{dW(t)}.$$

Particular case (stochastic salary and interest rate)

On the financial market we have two risky assets: (i) a rolling zero-coupon bond $B_K(t)$ with maturity K, and (ii) a stock whose price $S(t)$ follows a BGM. The two prices can be represented in the following way

$$\underbrace{\begin{bmatrix} \frac{dB_K(t)}{B_K(t)} \\ \frac{dS(t)}{S(t)} \end{bmatrix}}_{I_P^{-1}dP(t)} = \underbrace{\begin{bmatrix} r(t) - g(0,K)\sigma_r\xi_r \\ r(t) + \xi_r\sigma_{sr} + \xi_s\sigma_s \end{bmatrix}}_{\mu(t,z)} dt + \underbrace{\begin{bmatrix} -g(0,K)\sigma_r & 0 \\ \sigma_{sr} & \sigma_s \end{bmatrix}}_{\Sigma(t,z)} \underbrace{\begin{bmatrix} dW_r(t) \\ dW_s(t) \end{bmatrix}}_{dW(t)}.$$

The fund follows the SDE (the dependence on z has been omitted):

$$dX(t) = \big(X(t)\,r(t) + c(t) + w(t)'(\mu(t) - r(t)\mathbf{1})\big)\,dt + w(t)'\,\Sigma(t)\,dW(t). \tag{4}$$

Black and Scholes case (constant salary and interest rate)

Here we assume $s = 0$ and $n = 1$, no state variables, interest rate and contribution constant and positive: $r \geq 0$, $c \geq 0$. The Black and Scholes market consists of two assets, riskless with price $G(t)$:

$$dG(t) = rG(t)dt,$$

and risky asset, whose price $S(t)$ follows a GBM:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t).$$

The amount invested in the risky asset at time t is $w(t) \in \mathbb{R}$, and the fund grows according to

$$dX(t) = \left(X(t)r + w(t)(\mu - r) + c\right)dt + w(t)\sigma dW(t). \tag{5}$$

The mean-variance optimization problem

Pension fund member has a wealth x_0 at initial time t_0 and wants to maximize expected final wealth at retirement time T, while minimizing its variance:

$$\left[\mathcal{P}^{MV}_{t_0,x_0}\right] \qquad \sup_{w\in\mathcal{U}} J^{MV}(t_0,x_0,w) = \sup_{w\in\mathcal{U}} \left\{\mathbb{E}_{t_0,x_0}[X^w(T)] - \alpha\mathbb{V}_{t_0,x_0}[X^w(T)]\right\}, \tag{6}$$

where $X^w(t)$ is fund at time t when the control strategy w is applied
$\mathbb{E}_{t_0,x_0}(Z)$ is expectation at time t_0 with wealth x_0 of Z
$\mathbb{V}_{t_0,x_0}(Z)$ is variance at time t_0 with wealth x_0 of Z
$\alpha > 0$ measures risk aversion

Three approaches

The mean-variance problem is time-inconsistent!

There are 3 approaches to attack a time inconsistent problem (6):

1. precommitment ⇒ time inconsistent
2. dynamic optimality naive ⇒ time consistent
3. consistent planning ⇒ time consistent.

In this paper we focus only on the first and the second approaches. We have derived the *optimal investment strategy in closed form* for the two approaches in each financial market described before.

Simulations assumptions

We make Monte Carlo simulations in the Black and Scholes case. Assumptions are

- $x_0 = 1$
- $t_0 = 0$
- $T = 20$
- $c = 0.1$
- $r = 0.03$
- $\mu = 0.08, \sigma = 0.15 \Rightarrow \beta = 0.333$
- $K = 1.2 \Rightarrow \alpha = 5.056$
- weekly discretization, 1000 scenarios

Some Conclusions

- theoretical result: on average the wealth growth over time is exactly the same
- stand deviation of wealth lower with precommit than dyn opt naive
- in worst cases, precommitment wealth worse than dyn opt naive
- on average precommitment strategy less risky than dyn opt naive
- precommitment strategy highly more volatile than dyn opt naive